Optimizing athletic performance by influence curves

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FITZ-CLARKE, JOHN R., R. H. MORTON, AND E. W. BANISTER. *Optimizing athletic performance by influence curves.* J. Appl. Physiol. 71(3): 1151–1158, 1991.—Recent application of modeling techniques to physical training has opened the possibility for prediction from training. Solution of the inverse problem, determining a training program to produce a desired performance at a specific time, is also possible and may yield strategies for achieving better training and tapering (complete or relative rest for a period before competition) regimens for competitive athletes. A mathematical technique derived from model theory is described in this paper that allows the design of an optimal strategy of physical preparation for an individual to do well in a single future competitive event or cluster of events. Simulation results, using default parameters of a training model, suggest that presently accepted forms of taper for competition may remain too rigorous and short in duration to achieve the best result possible from the training undertaken.

training; taper

ACHIEVING OPTIMAL athletic performance requires an understanding of the effects of training during a competitive season so that strategies may be designed to place an athlete in peak condition at the exact time of competition. Training is still largely based on experience and intuition, but further improvement is possible if training effects can be quantified and optimized. A systems model for human performance (1) provides such a possibility and has been used to correlate training with performance under a wide variety of conditions with promising results. The basic assumption is that a training impulse w(t) (over time t), or dose of training, contributes to both fitness g(t) and fatigue h(t), and performance p(t) is related to the difference between these two quantities at any point in time. The detailed mathematics of this model are described elsewhere (6).

This paper presents a convenient technique for studying the inverse problem: given a desired performance p(t_p) for a competition at time t_p, what training program, defined from w(t), frequency of training, and characteristics of the taper procedure will achieve this result? More importantly, how may performance be maximized at any future time given the previous training history? To answer these questions, an influence curve technique is defined that allows conceptualization of the effect of each consecutive day's training on subsequent performance. The method may be used to design an optimal training strategy for a single performance or for several events in a competitive season.

Glossary

Dose	Alternative expression for quantity of training w(t) absorbed in a single training session, arbitrary units
g(t)	Hypothesized model component of performance ability, termed fitness, calculated from quantity of training undertaken, arbitrary units
h(t)	Hypothesized model component of performance ability, termed fatigue, calculated from quantity of training w(t) undertaken, arbitrary units
I(μ)	An integral function used to calculate performance after a period of uniform training, arbitrary units
k_1	Arbitrary weighting factor for fitness, dimensionless (initially 1)
k_2	Arbitrary weighting factor for fatigue, dimensionless (initially 2)
L(μ)	Influence curve ordinate that multiplies individual training impulse value w_i to give a contribution to performance at a future time when μ = 0, dimensionless
p(t)	Model-predicted performance determined from difference between weighted-model fitness k_1g(t) and weighted-model fatigue k_2h(t) at any time t during a training program, arbitrary units
t_g	Time measured before performance when training has maximum benefit at t_p, and influence curve has maximum ordinate here, days
t_k	Time period from cessation of training to peak performance, days
t_n	Time measured before performance within which training contributes more to fatigue than to fitness, days
t_p	Time of specific performance from specific time of onset of training, days
t_s	Interval of uniform training days followed by immediate cessation, days
w_s	Standard training impulse, arbitrary units
w(t)	Assessment of amount of training undertaken during a training session, also defined as a training impulse or dose and calculated as product of time (in min) spent training and change in heart rate ratio, arbitrary units
μ	Time interval measured before time of performance t_p, after a period of training, days

τ_1 Time constant determining time course of decay in accumulated fitness g(t) between training sessions, days

τ_2 Time constant determining time course of decay in accumulated fatigue h(t) between training sessions, days

INFLUENCE CURVES

An influence curve is a map or template showing how a function, distributed over a domain, affects a response at a specific point. For example, training may be considered as a dose distributed over the time domain resulting in performance at some future point in time. The influence curve is, by definition, the line representing the effect of a unit training impulse at any general time t on performance at a specific future time t_p. Thus, in a model of the effect of training on performance previously described for college swimmers (3), moderately trained runners (6), and more recently for elite weight lifters (2), after allocating a portion of a training impulse, defined from daily training, through multipliers k_1 and k_2, to represent elements of performance p(t) termed fitness g(t) and fatigue h(t), each element is allowed to decay with an appropriate time constant. Performance at a time t_p is then arbitrarily defined as the difference between the sum of residuals of each element from each day of training at t_p. This procedure is shown diagramatically in Fig. 1. Thus from the general equation in Ref. 3, defining performance p(t), performance at a specific future time t_p is

$$\begin{aligned} p(t_p) &= k_1 g(t_p) - k_2 h(t_p) \\ &= \int_0^{t_p} [k_1 e^{-(t_p-t)/\tau_1} - k_2 e^{-(t_p-t)/\tau_2}] w(t) dt \quad (1) \\ &= \int_0^{t_p} L(\mu) w(t) dt \end{aligned}$$

and the influence curve is simply

$$L(\mu) = k_1 e^{-\mu/\tau_1} - k_2 e^{-\mu/\tau_2} \quad (2)$$

where $\mu = t_p - t$ is time before performance, measured backward from t_p; t is the time for which training is continued; and k_1 and k_2 are the fitness and fatigue multipliers, respectively, as defined in Ref. 6. This curve is also identical to the time course of performance that would ensue from a single-unit training impulse. In the present case the stimulus is the daily training impulse w(t) assessed both from the intensity of heart rate response to training and the duration of a session. The dimensionless impulse response L(μ) transposed about the vertical axis graphically illustrates both the positive and the negative contribution of each day's training from the start of a program to the point of competition t_p. For a competition at time t_p, performance p(t_p) is determined by multiplying the training impulse w(t) by the influence curve L(μ) to obtain a product function, the area of which represents the performance at t_p. This is essentially a graphical representation of *Eq. 1*. In practical situations, w(t) is considered to be a series of impulses each day, rather than a continuous function, in which case the integral above becomes a summation where Δt = 1 day and

$$\begin{aligned} &= \sum_{i=1}^{j} [k_1 e^{-(j-i)/\tau_1} - k_2 e^{-(j-i)/\tau_2}] w_i \Delta t \\ &= \sum_{i=1}^{j} L(\mu_i) w_i \Delta t \end{aligned} \quad (3)$$

The function L(μ) need only be evaluated once for a current set of model parameters τ_1, τ_2, k_1, and k_2 and shows the influence of each increment of training on performance at t_p and therefore serves as a useful template for optimal placement of training w(t). Figure 1 shows an example of p(t_p) determined by the influence curve. Notice that L(μ) is constant, is assumed to be independent of training, and is dependent only on the four individual-specific model parameters estimated for an individual by regression of a predicted performance against criterion performance as described previously (6). Useful default parameters from which to begin this individual-specific iterative modeling have been found for several athletes (2) to be given by τ_1 = 45 days, τ_2 = 15 days, k_1 = 1, and k_2 = 2. Taking this approach, several observations are immediately apparent from the model proposed in Ref. 6 as a consequence of the mathematical theory advanced here.

Critical time frame for rest or reduced training before competition. Only training done earlier than a critical time before competition, defined as t_n, has a positive benefit on performance at t_p (although performances at later times may benefit, see Fig. 4). Training within t_n days before competition will contribute more to fatigue than to fitness and logically should be avoided. This critical point is given by the time when the increment to fatigue begins to exceed that to fitness, i.e., when $k_1 g(t_n) = k_2 h(t_n)$ for the unit training impulse case, or, using the default parameters proposed above, τ_1 = 45, τ_2 = 15, k_1 = 1, and k_2 = 2

$$\mu = \frac{\tau_1 \tau_2}{\tau_1 - \tau_2} \ln \frac{k_2}{k_1} = 16 \text{ days} \quad (4)$$

so that t_n is 16 days before t_p.

Thus t_n depends directly on individual-specific model parameters estimated from modeling-predicted performance, measured from training, against real performance responses (6). Table 2 shows how t_n may range between 15.8 ± 6.5. At its higher end (~23 days), t_n accords reasonably closely to a period, not of complete rest, but of reduced training before competition described for elite swimmers (21 days) (7) and to the time taken to achieve optimal performance (30 days) on endurance tests in moderately trained young men and women who trained for 10 wk and then reduced their training by 70% for a further 15 wk (5).

Equation 4 is analogous to *Eq. 10* in Ref. 6. However, it differs slightly, as the latter was derived for the case of a single impulse each day rather than the more general continuous training function assumed here.

Time period before competition about which training is maximally beneficial. The greatest benefit, again using the default parameters 45, 15, 1, and 2 for τ_1, τ_2, k_1, and

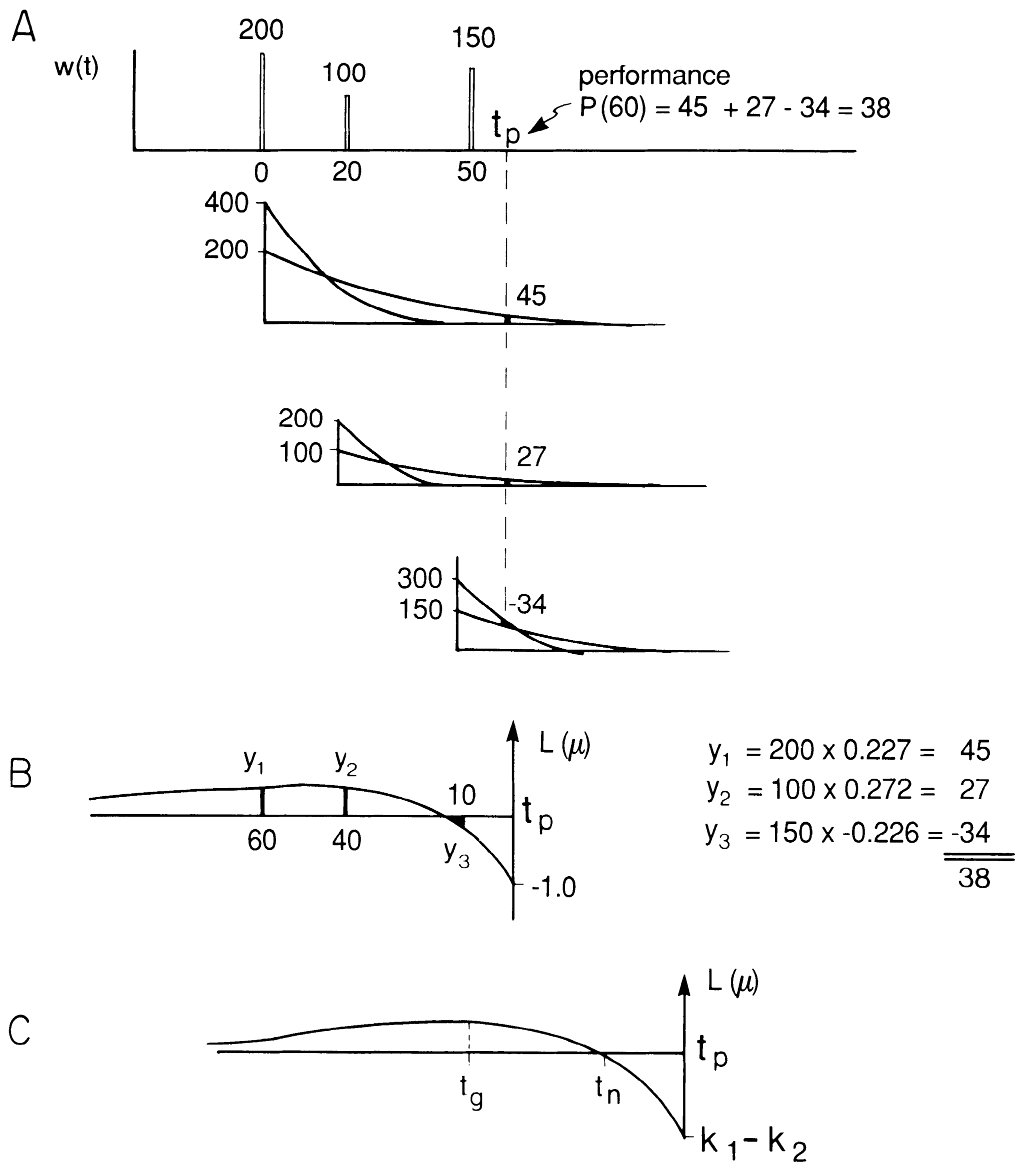

FIG. 1. *A*: performance p(t) may be considered as summation of residuals of contribution of each day's training impulse decayed to performance time as in this example of training for interval t_s (equal to 0, 20, and 50 days) for performance p(t) on day t_p = 60. Note that a training impulse of 200 units results in increments to fitness [w(t) × k_1] and fatigue [w(t) × k_2] of 400 and 200 units, respectively, on day t = 0. These values decay exponentially to p(t_p) as shown. Each subsequent training impulse likewise adds a contribution according to its initial magnitude. Contribution of each training impulse to p(60) is shown by black area between curves and is negative when performance occurs before fatigue has decayed to 0. *B*: same result may be calculated with more insight using a single influence curve, which shows relative contribution of each training impulse to performance at single specific future time. Right-hand origin of influence curve [i.e., dimensionless ordinate of L(μ) extending from negative 1.0 when k_2 = 2 and k_1 = 1] is placed at point where optimal performance is desired and relative contribution of each training impulse is immediately clear. Note detrimental effect of last training session (black area), inasmuch as it is in negative region of influence curve. *C*: training at t_g has greatest benefit to performance at t_p, whereas training done during interval between t_n and the t_p will be detrimental to performance at t_p. Shape of this weighting (influence) curve depends on model parameters τ_1, τ_2, k_1, and k_2. See *Glossary* for definitions of abbreviations.

k_2, respectively, is derived from training performed at a time where the influence curve is maximally positive. This is the time before competition when dL/dμ = 0 and

$$\mu = \frac{\tau_1 \tau_2}{\tau_1 - \tau_2} \ln\left(\frac{k_2}{k_1}\frac{\tau_1}{\tau_2}\right) = 40 \text{ days} \qquad (5)$$

so that t_g is 40 days before t_p.

It is especially interesting that the model recommends no training be done within ~16 days of the competition. This radical proposal is a practice not usually followed by athletes. Few athletes would be willing to break from their training for such a long period of time, although a similar period of gradual reduction seems acceptable (7). In addition, as previously noted, the values of both t_n and t_g depend critically on individually modeled parameters determined from training and may vary quite widely as estimated and shown in Table 2. The swimmers noted in Ref. 7 reduced training gradually from 9,000 yd/day, 5 days/wk for 3 wk, to 3,000 yd/day for 1–3 days/wk. In addition, it has been observed in elite runners that reducing training volume from a normal baseline level by 70% and reducing frequency of training by 17% improved

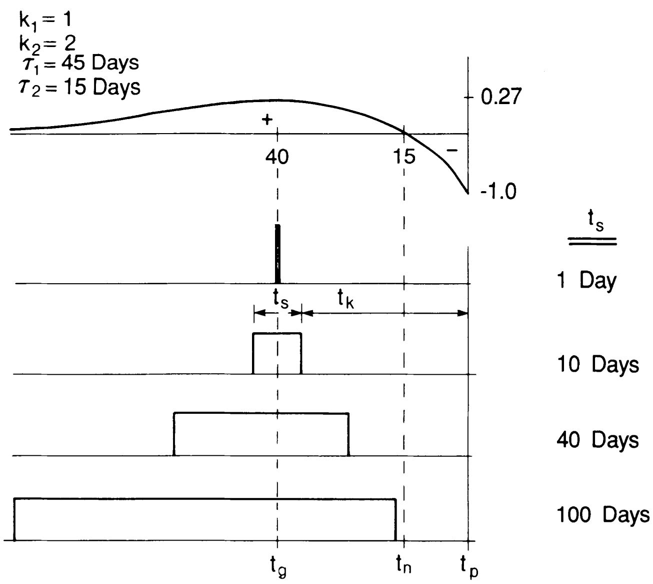

FIG. 2. Peak performance for a number of days of uniform training t_s is achieved when this training is placed optimally about maximum ordinate of influence curve t_g. Time to peaking t_k shortens as duration of training t_s is longer. Influence curve shown here is calculated using parameters shown at *top*.

group mean treadmill running time significantly by 0.5 min after 3 wk of reduced training. Group mean performance over 5,000 m also improved by nearly 5 s after 2 wk of reduced training (5).

TRAINING STRATEGIES

Single performance. We consider next the case of how to maximize performance if an athlete trains at a constant intensity for t_s successive days and then stops (Fig. 2). The influence line shows how this training should be placed at the high plateau around t_g in such a way as to maximize the area when $w(t)$ is multiplied by $L(\mu)$. Some examples for identical training each day are shown in Fig. 2. It should be noted how the time between termination of training and peak performance (t_k) becomes shorter as the training time t_s is longer. In fact, the examples shown here are exactly analogous to those considered in Fig. 4 of Ref. 6. Here, however, the influence line method has been used to achieve the same result with more insight.

Performance at any arbitrary time may be calculated from *Eq. 3*. However, for the special case that uniform training of magnitude w_s is maintained for t_s days and stopped t_k days before competition, the integral (I) of *Eq. 1* may be evaluated directly between the limits representing the duration of training, as shown in the APPENDIX. Performance at the time of competition for this special case may then be calculated as a difference between the integrated function at the start and end of a uniform training segment

$$p(t_p) = I_{(t_k)} - I_{(t_s + t_k)}$$

where *(6)*

$$I(\mu) = (k_1 \tau_1 e^{-\mu/\tau_1} - k_2 \tau_2 e^{-\mu/\tau_2}) w_s$$

Performance will be optimal only if t_k is chosen appropriately. As t_s becomes very long the optimal t_k will approach t_n. For example, if t_s = 60 days of training at a uniform w_s = 100 arbitrary units are followed by t_k = 20 days of rest, then for τ_1 = 45, τ_2 = 15, k_1 = 1, and k_2 = 2, the start and end of the training segment correspond to $\mu = t_k + t_s$ = 80 days and $\mu = t_k$ = 20 days before competition, respectively, and therefore from *Eq. 6*

$$I(20) = 2{,}095 \text{ and } I(80) = 746$$

$$P(t_p) = I(20) - I(80) = 1{,}349 \text{ arbitrary units}$$

The above example has analyzed performance only at a specific arbitrary time, saying nothing about the actual varying time course of performance $p(t)$. However, by imagining the influence curve $L(\mu)$ to be a movable template, performance may be advanced in time as illustrated in Fig. 3 (*top*). The origin of the template is placed at any time where $p(t)$ is desired, and succeeding performances at t_1, t_2, t_3, and t_p are given by the net value of the shaded areas (positive and negative) shown at the specific times of competitive events. In this way, the influence of a multiple-segment training regimen on performance at any given time becomes immediately apparent (Fig. 3, *bottom*). Alternatively $p(t)$ may be calculated directly by the explicit mathematical equations in Ref. 6

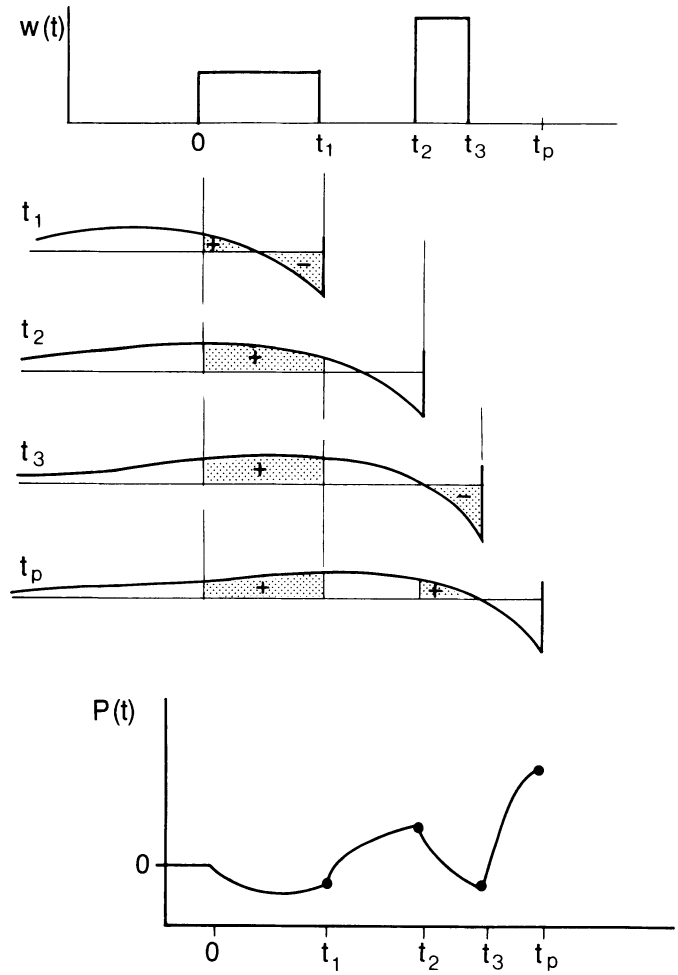

FIG. 3. *Top*: influence curve may be used as a moving template to track performance in time (t_1, t_2, t_3, t_p). Origin is placed at point where performance is desired (successively from t_1 through t_p), and net (positive plus negative) hatched area of influence curve shows relative contribution of training $w(t)$ to performance $p(t)$. Note that perfomance at time t_p is highest because all training segments fall completely within positive region of influence curve for performance at time t_p (*bottom*).

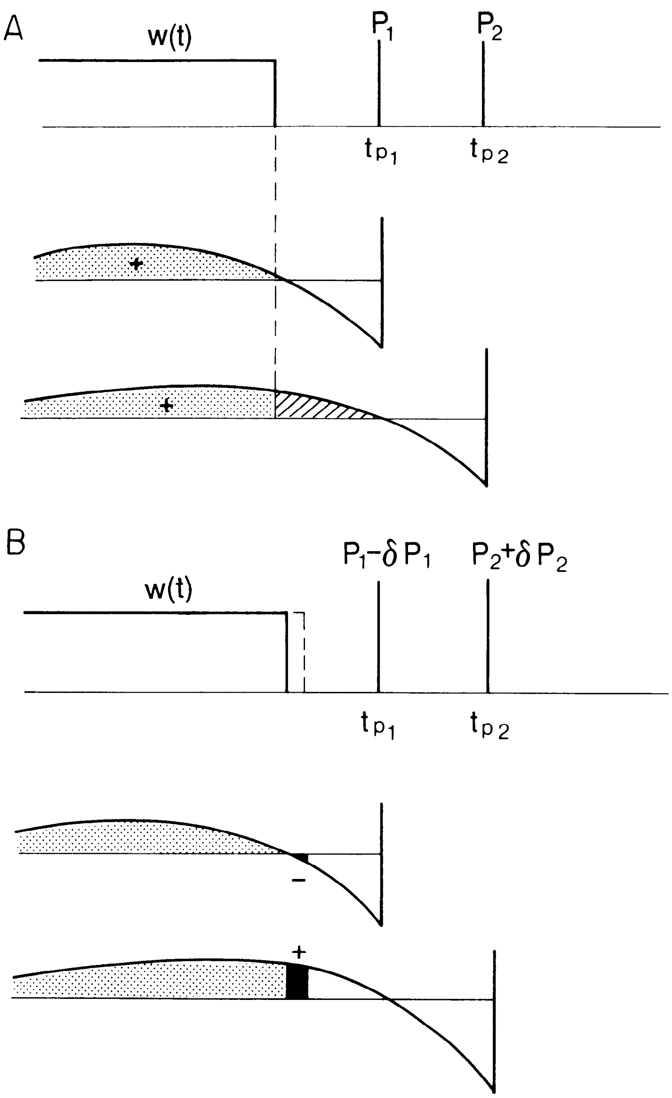

FIG. 4. *A*: theoretically optimizing training for 1st event at t_{p1} requires a recovery/taper period, either entirely without or with relatively little training, for 16 days before performance. This abstention represents lost training for a subsequent event at t_{p2}, as represented by the obliquely hatched area. *B*: alternatively, training up to dashed line of w(t) into negative region of influence curve for 1st event, which is represented by black shaded area, will reduce performance by δP_1 at t_{p1} but will improve 2nd performance by δP_2 at t_{p2} by a larger amount, i.e., by black shaded positive area around optimal ordinate of influence curve for performance at t_{p2}.

and in *Eq. 6* above; however, the graphical representation provides a clear conceptual picture of a preparation strategy, unavailable in the explicit calculation.

Several performances. The model predicts that an ultimate performance may only be achieved once in a season, since peaking requires a period of rest t_n before competition, which represents lost training for any subsequent events. Optimizing performance in more than one event, therefore, demands a compromise. The influence curve shows how such a compromise might best be met. In Fig. 4*A*, two equally important events are scheduled arbitrarily 16 days apart. In optimizing for the first event (P_1), using default parameters 45, 15, 1, and 2 for τ_1, τ_2, k_1, and k_2, respectively, t_n (= 16 days) of training are lost for the next event (P_2) if the negative effect of training too close to t_{p1} is to be avoided. The cost of this lost opportunity of training for P_2 is given by the hatched area and happens to occur where training would be highly beneficial for the second event, i.e., in the positive hatched area around t_g (Fig. 4*A, bottom*). In this particular case, training for the second event can unfortunately only be done best immediately before the first event. Worse still, more events placed 2 wk apart thereafter do not permit any training between events without some detrimental effect on these performances, depending on the individual's model parameters. A more likely strategy would be to train into the negative area of L(μ) a little (dashed line area) to benefit subsequent performances at some cost to the immediate one, as in Fig. 4*B* where the positive effect for t_{p2} of training in the dashed line area impinges into the negative, detrimental area for performance at t_{p1}. This depletes optimal performance from training by δP_1 (black negative area) but thereby enhances performance at P_2 by δP_2 by virtue of the extra training about t_g undertaken (black positive area).

The effect of training for several events may also be examined. Performance in each subsequent event is less than in the previous event, since the rest gaps before each competition represent cumulative lost training for future events. Figure 5 shows serial performance values (P_1, P_2, P_3, P_4) in arbitrary units when several events, 10 days apart, are each in turn maximized at some training cost to the others. If it is assumed that tapering for 16 days should be allowed to optimize performance for an event, then the event immediately before the one optimized can be allowed only 6 days of taper, and each competition two events or more before the one optimized can be allowed no taper at all before competition. These curves (Fig. 5, *bottom*) represent particular cases of training compromise, here shown to produce an increasing performance up to P_4 (along the path of the dashed line). Although it is clear that peaking may be designed to occur wherever desired, the best performance of all must necessarily be the one for which the longest and optimal training may be undertaken. Therefore, according to this model, if the athlete is out to break a record, it should be the last of a schedule of events, when all previous competitions have been compromised accordingly. Likewise, a best performance would be less likely early in the season, because less time has been devoted to training and the fitness time constant is relatively long. For example, peaking for an early event may compromise a later important event, such as at the Olympic games. Whether or not optimal performance may best be achieved late in the season at the cost of earlier events is an issue with important implications.

It must again be emphasized that in these simulations we have used the default coefficients $\tau_1 = 45$, $\tau_2 = 15$, $k_1 = 1$, and $k_2 = 2$, which we have found useful in making rough initial fits of predicted performance to real performance measures in the modeling process. They cannot be used to produce an accurate, general optimization schedule for a spectrum of individuals of different ability in a variety of events or even for a single individual at different stages of training. The essence of the procedure described for optimizing training is that it first depends on serial study of an individual's response to a continuing training stimulus, as described previously (6). Because the latter method is continuously interactive with an indi-

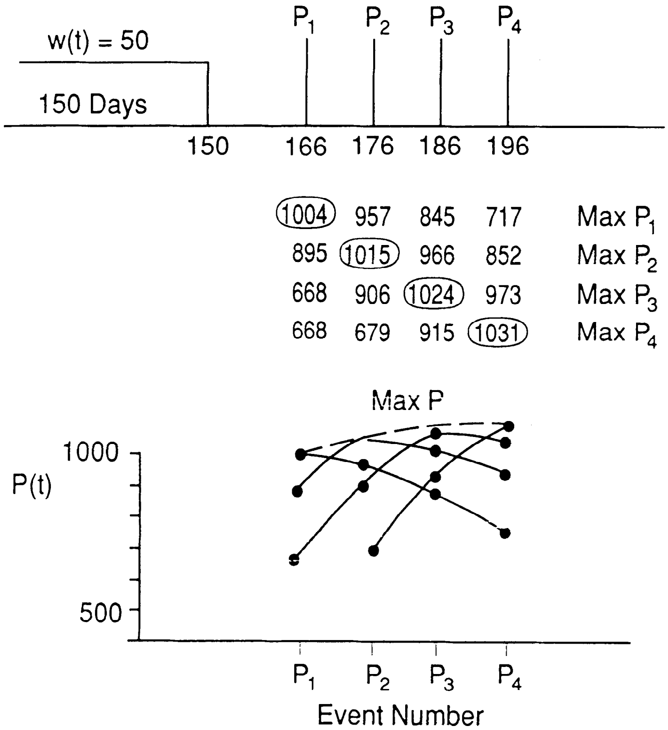

FIG. 5. Effect of training on successive hypothetical competition performances $<t_n$ days apart within restricted period. It is assumed that performance takes place after 150 days of training (t_s = 150) at constant w(t) of 50 arbitrary units. Optimizing performance for a single event among several subsequent events of equal importance involves training fully up to t_n (in this case = 16) days before each competition P_1, P_2, etc. Thus each event shown has been optimized in turn, and each relative peak performance (in arbitrary units) is circled. Nonoptimal performances, at other event times resulting from optimizing performance for a chosen event, are also shown in each row. These values may be calculated from *Eq. 3* or more directly from *Eq. 6*. For example, to maximize P_1, training should stop on *day 150* (so that $\mu = t_k = 16$) and a peak value of 1,004 will occur on *day 166* (t_{p1}). Likewise, maximum performance at P_2 occurs if training is carried on to *day 160* ($\mu = t_k = 16$ for t_{p2} and $\mu = t_k = 6$ for t_{p1}) and is then stopped. Performances resulting from this 2nd procedure are shown in 2nd row. On the other hand, a value of 668 units results for P_1 if training is carried on right up to day of competition, as is required in a strategy optimizing training for P_3 (t_{p3}) or P_4 (t_{p4}). P_1 remains unchanged at this value as long as continuous training through P_1 (t_{p1}) persists in preparation for 1 of other events.

vidual's state of preparation, the model parameters are always current. Specific model coefficients estimated for an individual in this way are probably only viable for predictive periods no longer than 60 days.

Sensitivity of t_n and t_g to model parameter values. The sensitivity of t_n and t_g to variations in the model parameters τ_1, τ_2, k_1, and k_2 is shown in Table 1. Nominal values of 45, 15, 1, and 2 have been used, respectively, for the four model input parameters. Table 1 shows the effect of increasing and decreasing each parameter in turn by 10%. Variations in τ_1 may be seen to have only a minor effect on t_n and t_g, whereas variations in τ_2, k_1, and k_2 cause changes that are somewhat greater in relative proportion.

Table 2 shows the effect on t_n and t_g of varying all four input parameters (τ_1, τ_2, k_1, k_2) together. The four parameters were assumed to be normally distributed with means chosen as nominal values used in previous studies (1–3, 6) from which individually modeled values have deviated only moderately. Standard deviations have been arbitrarily estimated for them to show a reasonably large variation in a hypothetical population. A Monte Carlo simulation produced an effect in which 200 sets of randomized model parameter values were generated by computer such that they had the statistical distribution shown in Table 1. Values of t_n and t_g were then calculated independently for each set, and the overall distribution was determined. The calculated coefficients of variation for t_n and t_g in Table 2 demonstrate that *Eqs. 4* and *5* can have a significant effect in magnifying the variance of these two quantities and emphasize the importance of using individual-specific model parameters in their evaluation.

The most interesting conclusion from this sensitivity analysis is that the nontraining, absolute rest interval t_n predicted from model parameters is considerably longer than any absolute rest period that we have been able to determine is used in practice. This remains true even when considerable individual variation is taken into account. However, as noted, rest as practiced by athletes is only relative, and an extreme reduction both in daily training and the frequency of training for an athlete may provide the equivalent optimal preparation as complete abstinence would provide in less well-trained individuals. The similarity of periods of reduced training (7–35 days) reported in the literature (4, 5, 7) and the range of values estimated for t_n, which produce an increment in physical performance on a standardized test, for wide ranging ability groups of male and female trainees is compelling.

TABLE 1. *Sensitivity of t_n and t_g to changes in model parameters*

	τ_1	τ_2	k_1	k_2	t_n	t_g
Nominal value	45	15	1.0	2.0	15.6	40.3
%Change	+10				−4.3	+0.7
	−10				+5.9	−0.3
		+10			+15.8	+9.6
		−10			−14.3	−9.2
			+10		−13.8	−5.3
			−10		+15.2	+5.9
				+10	+13.8	+5.3
				−10	−15.2	−5.9

See *Glossary* for definitions of abbreviations.

CONCLUSION

The method of influence lines provides a useful tool to study, conceptually, the effect of training on performance at any given time. Although used with exponential functions in this case, the method is general and is valid for any other weighting function L(μ) so long as performance represented by p(t) is linear with training w(t).

TABLE 2. *Effect of model parameter variabilities on t_n and t_g as determined by Monte Carlo simulation*

	τ_1	τ_2	k_1	k_2	t_n	t_g
Mean	45	15	1.0	2.0	15.8	40.1
SD	7.0	2.5	0.1	0.3	6.5	8.3
%CV	15.6	16.7	10.0	15.0	41.1	21

Parameter SD values were chosen as described in text. CV, coefficient of variation. See *Glossary* for definitions of abbreviations.

This is not a severe limitation, for if training does in fact have a nonlinear effect according to the training impulse definition, this is accounted for to some degree when the performance function $p(t)$ is converted to a real measure of performance, such as an actual time, by some empirical conversion equation.

As has been seen using assumed model constants from previous studies (1, 3, 6), it is predicted that an athlete should terminate training ~16 days before competition. Most athletes would consider this far too long and would prefer to taper until almost to the time of competition. If, on the other hand, the model's prediction has true validity, it represents an opportunity to produce even better performance.

Experiments are needed to test the model further under controlled protocols. In particular, work is needed to study the physiological processes occurring during the 2-wk period before competition to determine the positive and negative aspects of training during this time. It is here that prediction and current practice are in maximum disagreement.

What are the specific time constant values and contributions of separate cardiorespiratory, cardiovascular, muscular, and motor coordination components to overall performance? For example, the training impulse by definition reflects primarily cardiorespiratory effort; however, strength may have a different set of constants, and some evidence from elite weightlifters already suggests this (2). How quickly do motor patterns decay, and to what extent might they contribute? The model attempts to incorporate these components into the four empirical parameters; however, further improvement is possible if the components themselves can be isolated. Furthermore, why do time constants vary between individuals? How is this reflected by such factors as the muscle fiber-type ratio, biochemical indexes, or other performance-determining parameter? What is the role of light voluminous training versus short intense training? The multiplying factor in the definition of the training impulse in the original model (6) attempts to account for this difference. However, because typical training regimens involve patterns of varying intensities, it becomes apparent how crucial the nature of this adjustment really is in practical terms.

Some of these questions are clearly difficult to answer at present. However, controlled experiments may be designed to isolate some of these factors in a way not possible with an athlete following an intuitive rather than a rigidly controlled program. Improvement of the model and design of optimal training strategies will no doubt be directly related to the number and success of controlled experiments designed to answer these fundamental questions.

APPENDIX

Derivation of $p(t_p)$ as an Integral

In Ref. 6, performance $p(t)$ was defined as the difference between fitness $g(t)$ and fatigue $h(t)$

$$p(t) = k_1 g(t) - k_2 h(t) \qquad (A1)$$

where $g(t)$ and $h(t)$ were evaluated as iterated functions of discrete training impulses $w_i(t)$. In the general case, the dose need not be an impulse but may have any arbitrary continuous history $w(t)$. At a specific future time t_p, fitness $g(t_p)$ and fatigue $h(t_p)$ are the cumulative result of a series of infinitely small doses, each decayed exponentially during the time from when the incremental dose $w(t)dt$ occurred to the time when its effect is desired, t_p. Therefore the total effect of all doses may be expressed as an integral of all previous time

$$g(t_p) = \int_0^{t_p} [e^{-(t_p-t)/\tau_1}]w(t)dt$$

$$h(t_p) = \int_0^{t_p} [e^{-(t_p-t)/\tau_2}]w(t)dt$$

Some readers will recognize these as convolution integrals. Performance $p(t_p)$ follows from *Eq. A1* as

$$p(t_p) = \int_0^{t_p} [k_1 e^{-(t_p-t)/\tau_1} - k_2 e^{-(t_p-t)/\tau_2}]w(t)dt \qquad (A2)$$

Case of Uniform Training

If uniform training, $w(t) = w_s$, is carried out for a period of $t = t_s$ days followed by t_k days of rest before performance of t_p so that $t_p = t_s + t_k$, then the integral (*Eq. A2*) may be evaluated directly, since w_s is constant, between the limits of $t = 0$, the onset of training, and $t = t_s$, the number of days of training completed

$$\begin{aligned} p(t_p) &= w_s \int_0^{t_s} [k_1 e^{-(t_p-t)/\tau_1} - k_2 e^{-(t_p-t)/\tau_2}]dt \\ &= w_s\, [-k_1\tau_1 e^{-(t_p-t)/\tau_1} + k_2\tau_2 e^{-(t_p-t)/\tau_2}]_0^{t_s} \end{aligned} \qquad (A3)$$

If $p(t_p)$ is expressed in terms of the time previous to performance when training ceases (μ), then

$$\mu = t_p - t$$

$$d\mu = -dt$$

and *Eq. A3* becomes

$$\begin{aligned} p(t_p) &= w_s \int_{t_s+t_k}^{t_k} [-k_1 e^{-\mu/\tau_1} + k_2 e^{-\mu/\tau_2}]d\mu \\ &= w_s[k_1\tau_1^{-\mu/\tau_1} - k_2\tau_2 e^{-\mu/\tau_2}]_{t_s+t_k}^{t_k} \end{aligned} \qquad (A4)$$

since when $t = 0$, $\mu = t_p = t_s + t_k$ and when $t = t_s$, $\mu = t_p - t_s = t_k$. Thus

$$p(t_p) = I_{(t_k)} - I_{(t_s+t_k)}$$

where

$$I(\mu) = [k_1\tau_1 e^{-\mu/\tau_1} - k_2\tau_2 e^{-\mu/\tau_2}]w_s$$

Derivation of t_n and t_g

The critical time t_n occurs when $L(\mu) = 0$

$$L(\mu) = k_1 e^{-\mu/\tau_1} - k_2 e^{-\mu/\tau_2}$$

$$\frac{k_2}{k_1} = e^{-\mu(1/\tau_1 - 1/\tau_2)}$$

$$\ln\frac{k_2}{k_1} = \frac{\mu(\tau_1 - \tau_2)}{\tau_1\tau_2}$$

$$\mu = \frac{\tau_1\tau_2}{\tau_1 - \tau_2}\ln\frac{k_2}{k_1}$$

This interval corresponds to a time t_n before competition at t_p. Training within t_n days before t_p begins to have a negative impact on $p(t_p)$.

The maximum training benefit to performance occurs when $L(\mu)$ is greatest, or $dL/d\mu = 0$

$$\frac{dL}{d\mu} = -\frac{k_1 e^{-\mu/\tau_1}}{\tau_1} + \frac{k_2 e^{-\mu/\tau_2}}{\tau_2} = 0$$

$$\frac{k_2\tau_1}{k_1\tau_2} = e^{-\mu(1/\tau_1 - 1/\tau_2)}$$

$$\mu = \frac{\tau_1\tau_2}{\tau_1 - \tau_2} \ln \frac{k_2\tau_1}{k_1\tau_2}$$

This interval corresponds to a time t_g, before competition at t_p, about which training is maximally beneficial for performance at t_p.

NOTE ADDED IN PROOF

Since this paper went to press, a paper by Koutedakis et al. (*Br. J. Sport Med.* 24: 248–252, 1990), specifically indicating performance in overtrained "elite" athletes was improved significantly by between 21 and 35 days of complete rest, has come to the authors' attention.

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